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On the Brownian Motion of a Frequency-Modulated Oscillator

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The Langevin equations for a harmonically bound particle, where the force constant is a given function of time, is solved. The mean and mean-square properties of the solution are determined. The analysis is then specialized to the case of a force constant periodic in time. It is shown that the mean fluctuations remain bounded in time precisely when the mean motion is stable (remains bounded).

KEY WORDS: Langevin equations; Brownian motion; fluctuations; Hill's equation.

1. INTRODUCTION

The phenomenological theory of the Brownian motion for linear systems in time-independent environments is quite well understood. Details can be found, for example, in the well-known review articles of Wang Chang and Uhlenbeck[(1)] and Chandrasekhar.[(2)] The theory has been extended to cover the cases of non-Markovian noise[(3)] and fluctuating parameters,[(4)] e.g., the frequency of an oscillator. One case which has not been treated so far is the case when the parameters of the system, while remaining deterministic, are time dependent, due, perhaps to external forces imposed on the system. One can think, for example, of an *RLC* circuit in which the capacitance is periodically varied. The object of this paper is to show how such a system can be treated.

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The reason, one might conjecture, that this problem has not been considered is that it does not admit of a relaxation toward equilibrium. The action of time-dependent external forces effectively prevents such a relaxation. Nevertheless, systems with time-varying parameters are not rare, and it is interesting to study their fluctuations. Another reason for studying this problem is that the model we treat can also be looked on as the linearization of a nonlinear equation of motion about a periodic solution, with added noise. Such linearizations are interesting to study for the light they throw on the stability of the periodic solution of the original nonlinear equation.

In addition to the *RLC* circuit just mentioned, another physical realization of the problem to be studied is a pendulum with a forced motion of the point of support. Other electromechanical devices described, in some approximation, by this type of model are discussed by McLachlan.(5)

The model which we shall treat is very similar to the models used in the classical case. We consider a particle in an environment with time-independent linear friction, and a time-dependent linear external force, subject to Gaussian white noise. That is, we have to study the Langevin equation

$$\ddot{x} + \beta\dot{x} + f(t)x = F(t) \tag{1}$$

where β is the friction, $f(t)$ the external force, and $F(t)$ the random force. $F(t)$ is characterized by its statistical properties. It is a Gaussian random variable with

$$\langle F(t)\rangle = 0 \tag{2a}$$

$$\langle F(t)F(s)\rangle = mkT\beta\,\delta(t-s) \tag{2b}$$

For the sake of definiteness, we have assumed that the random force is independent of the time-dependent external forces. Therefore the coefficient of the delta function in (2b) is the same as that appropriate to a time-independent environment. In the present case, one does not have the fluctuation-dissipation theorem at one's disposal to determine this coefficient, as one does in the case of the time-independent environment.

We shall produce a formal solution to Eq. (1) and deduce its statistical properties with the help of Eqs. (2a) and (2b).

2. SOLUTION OF THE EQUATIONS

To solve Eq. (1), we first eliminate the first derivative term. Setting

$$x(t) = y(t)e^{-\beta t/2} \tag{3}$$

one finds that y satisfies

$$\ddot{y} + [f(t) - \beta^2/4]y = A(t), \qquad A(t) = F(t)\exp(\beta t/2) \tag{4}$$

Now let $\phi_1(t)$ and $\phi_2(t)$ be special solutions to the homogeneous form of Eq. (4), i.e., Eq. (4) with $A(t)$ set equal to zero. Specifically,

$$\begin{aligned} \phi_1(0) &= 0; \qquad \dot{\phi}_1(0) = 1 \\ \phi_2(0) &= 1; \qquad \dot{\phi}_2(0) = 0 \end{aligned} \tag{5}$$

Then the general solution to Eq. (4) is

$$y(t) = \int_0^t G(t,s)A(s)\,ds + A\phi_1(t) + B\phi_2(t) \tag{6}$$

where

$$G(t,s) = \phi_1(t)\phi_2(s) - \phi_2(t)\phi_1(s) \tag{7}$$

One can identify the constants A and B with $\dot{y}(0)$ and $y(0)$, respectively.

Making use of Eq. (2a), one sees immediately that $\langle y(t)\rangle$ is a solution of the homogeneous equation, i.e.,

$$\langle y(t)\rangle = y(0)\phi_2(t) + \dot{y}(0)\phi_1(t) \tag{8}$$

and $\langle y(t)\rangle$ is obtained immediately by differentiating (8). This is only to be expected, by the linearity of the equations. To get $\langle x(t)\rangle$ one merely multiplies by $\exp(-\beta t/2)$.

The mean square fluctuation in $y(t)$ may be computed, through (6), by

$$\begin{aligned} \langle(\delta y)^2\rangle &= \langle[y(t) - \langle y(t)\rangle]^2\rangle \\ &= \int_0^t ds \int_0^t ds'\, G(t,s)G(t,s')\langle A(s)A(s')\rangle \end{aligned} \tag{9}$$

By virtue of Eq. (2b), this can be written as

$$\langle(\delta y)^2\rangle = mkT\beta \int_0^t G^2(t,s)e^{\beta s}\,ds \tag{10}$$

In terms of the original variable x, one has

$$\langle(\delta x)^2\rangle = mkT\beta \int_0^t G^2(t,s)e^{-\beta(t-s)}\,ds \tag{11}$$

One can also write down similar expressions for correlation functions $\langle x(t)x(s)\rangle$, $\langle x(t)\dot{x}(s)\rangle$, etc. The method should be obvious from the foregoing, and we omit further details.

If one sets $f(t) = \text{const}$, one is back to the classical problem of the Brownian motion of a harmonically bound particle, and the well-known results are recovered in that limit.

The function $G(t,s)$ defined by Eq. (7) is the fundamental building block of our solutions. The solutions (5) from which G is constructed may not, on the other hand, be the most convenient set of solutions, e.g., they may not be the "standard," tabulated, linearly independent solutions of the homogeneous equation. However if we take any other linearly independent set ψ_1 and ψ_2 related to ϕ_1 and ϕ_2 by

$$(\psi_1, \psi_2) = \mathbf{A}(\phi_1, \phi_2) \tag{12}$$

where $\mathbf{A}$ is a nonsingular constant matrix, then

$$\psi_1(t)\psi_2(s) - \psi_2(t)\psi_1(s) = \Delta(\mathbf{A})\left[\phi_1(t)\phi_2(s) - \phi_2(t)\phi_1(s)\right] \tag{13}$$

where $\Delta(\mathbf{A})$ is the determinant of $\mathbf{A}$. Hence, one is free to use any pair of linearly independent solutions to construct the kernel G, provided one normalizes by the proper determinant.

Since $F(t)$ has been assumed to be Gaussian and, by (6), $y(t)$ is a linear function of F, one sees immediately that $y(t)$ is a Gaussian random variable with mean and variance given by Eqs. (8) and (10), respectively. Hence $x(t)$ is also a Gaussian random variable, with mean $\langle y(t)\rangle e^{-\beta t/2}$ and variance given by Eq. (11).

The stochastic process $x(t)$ is, however, no longer a stationary, or even wide-sense stationary, process. The correlation function $\langle x(t)x(s)\rangle$ is not a function of $t - s$ alone. In view of the externally imposed time dependence, this result is not surprising; the physical situation is not temporally homogeneous.

Without further specification of the forcing term $f(t)$, the problem is too generally defined for further progress to be possible. We therefore pass on to discuss a specific example.

3. AN EXAMPLE

A case of some interest is that in which the forcing term has a persistant nature. The simplest example of persistence is periodicity (constancy being a trivial kind of periodicity). The differential equation with periodic coefficients which has been most thoroughly studied is Mathieu's equation, but we shall choose a slightly more general example.

We shall consider Hill's equation, which means taking $f(t)$ in Eq. (1) to be a periodic function of t, with period π, which is even in t. The fixing of the period as π merely means choosing an appropriate unit of time. Mathieu's equation is a special case when $f(t)$ is taken as $f = a + 16q\cos 2t$ (using the notation of Whittaker and Watson[6]). In neither case can solutions be written down in closed form. The Mathieu functions, about which a great deal is known, are periodic solutions with period 2π, but only exist for special values of a; and they are of little interest here. Therefore, we shall not present detailed analytical formulas, but only present results on the general nature of the fluctuations.

The result which enables us to make some progress is Floquet's theorem, which states that, if one has a differential equation with periodic coefficients, of period π, say, it admits a solution of the form

$$x_1(t) = e^{\mu t} z(t) \tag{14}$$

where $z(t)$ is periodic, with period π. In the case of Hill's equation, since $f(t)$ is taken to be even, $x_2(t) = x_1(-t)$ is also a solution, which is linearly independent of $x_1(t)$ (since the ratio changes by $e^{2\mu\pi}$ when t changes by π).

The x_1 and x_2 are not necessarily the solutions ϕ_1 and ϕ_2 defined by Eq. (5), but are linear combinations of them. We can always choose μ so that Re $\mu \geqslant 0$, since $+\mu$ occurs in one solution, $-\mu$ in the other, so that we will always call x_1 the Floquet solution for which Re $\mu \geqslant 0$.

By Eq. (8), $\langle y(t)\rangle$ increases in time without bound unless Re $\mu = 0$. Hence $\langle x(t)\rangle$ increases without bound unless Re $\mu \leqslant \beta/2$. When the solution increases without bound, one says that the motion is unstable. Physically, something always happens to cut off this unbounded motion. Nonlinear terms, neglected in the formulation, become important, or there may be a back-reaction changing the external forces. In any event, the character of the problem is changed beyond the scope of the present analysis. Therefore, we confine ourselves to the so-called stable case where $\langle x(t)\rangle$ remains bounded. The question to be answered, then, is how do the fluctuations behave? Do they also remain bounded?

By virtue of Eq. (13), we may use x_1 and x_2 to construct $G(t,s)$. The value of $\Delta(\mathbf{A})$ is of no interest for the present question, since it is a constant. Then $G(t,s)$ has a growing term of the form $\exp[\mu(t-s)] \times$ a periodic function, and a decaying term of the form $\exp[-\mu(t-s)] \times$ a periodic function, if Re $\mu > 0$. If Re $\mu = 0$, then both terms are bounded. In this latter case, Eq. (11) shows immediately that $\langle(\delta x)^2\rangle$ is bounded.

In the former case, when Re $\mu > 0$, the only part of $G^2(t,s)$ that is growing behaves like $\exp[2\mu(t-s)] \times$ a periodic function. Thus, the growing part of the integrand in (11) contains the factor $\exp[(2\mu-\beta)(t-s)]$. But, since Re $\mu < \beta/2$, the exponent is negative, and the integral is bounded for all t. In the limiting case when μ is real and equal to $\beta/2$, the fluctuations diverge linearly in t, modulated by a periodic function. The case of the free particle is an example of this. The case of the harmonically bound particle is also a special case of the foregoing development.

Specializing to Mathieu's equation, it is possible to get more specific results by expanding in powers of q for small q. In the small-q limit, the primary effect is to renormalize the oscillator frequency. The results are little different from the case of the harmonically bound particle. Consequently we do not reproduce them here. In our opinion, the qualitative results depending only on the periodicity of $f(t)$ are much more interesting than these approximations.

4. DISCUSSION

We have determined the Brownian motion of an oscillator with a time-dependent frequency. The frequency is externally determined, but is not a random variable. The random force was assumed to be delta-correlated Gaussian, and the friction assumed to be constant, linear, and related to the random force through the fluctuation-dissipation theorem. That is, the fact that the system parameters are being modulated is assumed not to affect the systematic friction characteristic of the interaction of the system with its environment. If one thinks in electrical circuit terms, the capacitance is being modulated, while the noise source and dissipation take place in the resistance, so that the assumptions are not unreasonable physically.

The main results of this investigation are (i) Eqs. (6) and (11), which give the displacement and mean square fluctuation in terms of solutions to the deterministic equations, and (ii) the determination of the qualitative nature of the growth of the mean square fluctuation when the modulation is periodic.

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